Сети с ожиданием: регулярные оптимальные расписания

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Graph of the matrix over the tropic semifield

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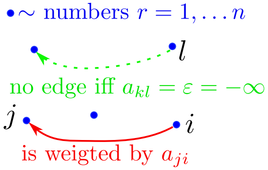

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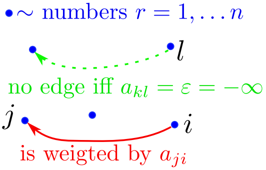

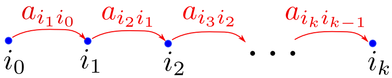

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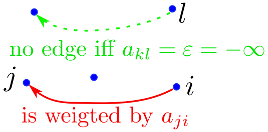

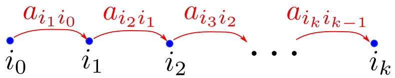

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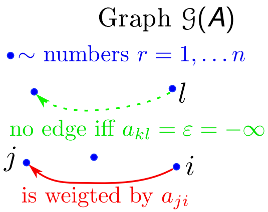

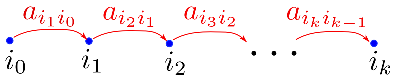

- Path p from i_0 to i_k of length $L(p) := k$
- The weight of the path

$$W(P) := a_{i_1 i_0} + a_{i_2 i_1} + a_{i_3 i_2} + \cdots + a_{i_k i_{k-1}}$$

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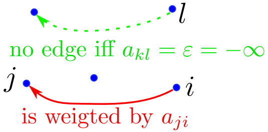

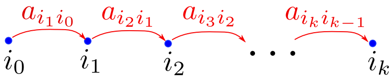

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$$W(P) := a_{i_1 i_0} + a_{i_2 i_1} + a_{i_3 i_2} + \cdots + a_{i_k i_{k-1}}$$
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Geometric interpretation of matrix degrees

Lemma

The element b_{ji}^k of the matrix A^k is the maximal weight of a path in $\mathcal{G}(A)$ that goes from node i to j and has a length of k ($:= \varepsilon$ is there is no such a path).

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Proof by induction on k.

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$$b_{ji}^{k+1} = [A \otimes A^k]_{ji}$$

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Corollary

The element c_{ji}^k of the matrix $E \oplus A \oplus A^2 \oplus \cdots \oplus A^k$ is the maximal weight of a path in $\mathcal{G}(A)$ that goes from node i to j and whose length $\leq k$ ($:= \varepsilon$ is there is no such a path).

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$$\mathcal{M}(C) = \frac{W(C_-) + W(C_+)}{L(C_-) + L(C_+)} = \sum_{\varsigma=\pm} \frac{L(C_\varsigma)}{L(C_-) + L(C_+)} \frac{W(C_\varsigma)}{L(C_\varsigma)} \leq m(A) \sum_{\varsigma=\pm} \frac{L(C_\varsigma)}{L(C_-) + L(C_+)}.$$

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$s = 0$ — clear

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for any $b \in \mathbb{R}_{\max}^n$, the element $x := A^* b$ is a solution for the equation

$$x = b \oplus Ax.$$

Spectral theory of square matrices

Definition

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x determines a regular schedule with departures every λ units of time

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$$\begin{array}{l} \lambda \otimes x_{i_0} = a_{i_0 i_1} \otimes x_{i_1} \\ \lambda \otimes x_{i_1} = a_{i_1 i_2} \otimes x_{i_2} \\ \lambda \otimes x_{i_2} = a_{i_2 i_3} \otimes x_{i_3} \\ \vdots \\ \lambda \otimes x_{i_{k-1}} = a_{i_{k-1} i_k} \otimes x_{i_k} \end{array} \quad \Bigg| \quad \Rightarrow \quad \begin{array}{l} \lambda^k \otimes \underbrace{x_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_{k-1}}}_{\chi} \\ = a_{i_0 i_1} \otimes a_{i_1 i_2} \otimes \cdots \otimes a_{i_{k-1} i_k} \\ \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_{i_k} \\ = [a_{i_0 i_1} + a_{i_1 i_2} + \cdots + a_{i_{k-1} i_k}] + \chi \end{array}$$

Existence and uniqueness of an eigenvalue

Theorem

Suppose that the graph $\mathcal{G}(A)$ is strongly connected. Then there exists an eigenvalue of A, this eigenvalue is unique and equal to the maximum cyclic mean $m(A)$.

Uniqueness:

$$Ax = \lambda x \Rightarrow A^k x = \lambda^k x \Rightarrow a_{ii}^k + x_i \leq k\lambda + x_i \Rightarrow a_{ii}^k \leq k\lambda$$

$\Rightarrow$ the weight of any cycle of length k does not exceed $k\lambda$

$\Rightarrow$ its mean weight does not exceed $\lambda \Rightarrow m(A) \leq \lambda$

let us pick i_0 there exists i_1 such that $\lambda \otimes x_{i_0} = a_{i_0 i_1} \otimes x_{i_1}$

there exists i_2 such that $\lambda \otimes x_{i_1} = a_{i_1 i_2} \otimes x_{i_2}$

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$\Rightarrow$ the weight of the cycle $W(C) = k\lambda$

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$\Rightarrow$ the weight of the cycle $W(C) = k\lambda \Rightarrow \mathcal{M}(C) = \lambda$

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$$\Rightarrow \begin{aligned} &\lambda^k \otimes \underbrace{x_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_{k-1}}}_{\chi} \\ &= a_{i_0 i_1} \otimes a_{i_1 i_2} \otimes \cdots \otimes a_{i_{k-1} i_k} \\ &\otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_{i_k} \\ &= [a_{i_0 i_1} + a_{i_1 i_2} + \cdots + a_{i_{k-1} i_k}] + \chi \end{aligned}$$

$\Rightarrow$ the weight of the cycle $W(C) = k\lambda \Rightarrow \mathcal{M}(C) = \lambda \Rightarrow m(A) \geq \lambda$